

On the onset of interference effects during the formation of the Bose-Einstein condensate

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Abstract

In this paper we derive the equations characterizing the boundary layer which describes the transition of the distribution function of a gas of weakly interacting bosons to the distribution function of the gas in the presence of a Bose-Einstein condensate. To this end, we first rederive the classical Uehling-Uhlenbeck equation taking as a starting point the dynamics of a system of many weakly interacting quantum particles. The solutions of the Uehling-Uhlenbeck equation yield blow-up in finite time. Near the blow-up time the approximations used to derive the Uehling-Uhlenbeck equation break down. We derive the set of equations that describe the building of correlations and the onset of quantum interference effects for the many-particle hamiltonian system under the assumption that the blow-up for the Uehling-Uhlenbeck equation takes place in a self-similar form.

Keywords: Bose Einstein condensation, Uehling-Uhlenbeck equation, boundary layer, blow up, self similar solutions.

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I. INTRODUCTION.

The kinetic equation that describes the evolution of a rarefied system of bosons was obtained in [21] by L. W. Nordheim and in [26] by E. A. Uehling and G. E. Uhlenbeck. In the limit of Born's approximation, the resulting equation, also known as quantum Boltzmann equation is the following:

$$\frac{\partial f}{\partial t} + \frac{p_1}{m} \nabla_x f(p_1, x, t) = C(f, f) \quad (1)$$

$$C(f, f) = \frac{4\pi g^2}{\hbar} \int \frac{dp_2}{(2\pi\hbar)^3} \int \frac{dp_3}{(2\pi\hbar)^3} \int \frac{dp_4}{(2\pi\hbar)^3} (2\pi\hbar)^3 \delta(\varepsilon(p_1) + \varepsilon(p_2) - \varepsilon(p_3) - \varepsilon(p_4)) \times \\ \times q[f](p_1, p_2; p_3, p_4, t) \quad (2)$$

$$q[f](p_1, p_2; p_3, p_4, t) = [f(p_3)f(p_4)(1 + f(p_1))(1 + f(p_2)) - \\ - f(p_1)f(p_2)(1 + f(p_3))(1 + f(p_4))] \quad (3)$$

with $g = 4\pi a\hbar^2/m$ where m is the mass of the particles, $\varepsilon(p) = |p|^2/2m$ is the energy of the particles and a is the s-wave scattering length (cf. [28] for example).

The starting point for the derivation of a kinetic system of equations is a set of equations describing the dynamics of a system of N particles. In the case of weakly interacting bosons such dynamics is described by the Schrödinger equation for a system of N interacting particles. Assuming that the interaction between different particles are weak enough it is possible to obtain suitable evolution equations for the one-particle distribution function using a perturbative method. For classical particles, this has been mathematically proved for short times (cf. [19]) or globally in time for special situations (cf. [14]). This is the standard method used to derive the Uehling-Uhlenbeck equation (see for example the classical monographies [1, 3], as well as the more mathematically oriented approaches in [5], [7], [8], and [24]). Similar arguments for the Fermionic case may be found in [13] and [12].

It turns out that the solutions of (1)-(3) can develop singularities in finite time, as it has been obtained in the numerical simulations for spatially homogeneous distributions of particles in [22], [23] as well as in [18]. The interpretation of this blow up phenomenon, given by the authors of these papers, is that such event corresponds to the formation of the B-E condensate. As we will see in this paper, the derivation of the U-U equation, taking as a starting point a quantum many particles system, is not valid near the time of the formation of the singularity.

On the other hand, the quantum dynamics of the particles in the condensate is described by the Gross Pitaevskii equation (cf. [2, 10, 25, 28]). A rigorous mathematical proof of the precise formulation of this fact has been obtained in [20] for the stationary case and in [9] for the non equilibrium case with short range interactions in suitable scaling limits.

We are interested in the process formation of the condensate, a question which has already been considered by several authors, see for example [6], [16], [25]. Our main goal is to describe in a detailed manner how the transition between the kinetic regime described by the U-U equation, and the quantum regime described by means of a nonlinear Schrödinger equation. The relevant non-dimensional parameter is the quotient $\Delta E \Delta t / \hbar$, where ΔE is a characteristic value of the energy and ΔT is a characteristic time scale for the variation of the density distributions. Interference effects cannot be ignored if this non-dimensional number becomes of order one. We derive then the equation of the boundary layer which should describe in detail the transition from the distribution function at the critical time to the density function in which the condensate is present.

Part of the arguments presented in this paper are standard in statistical physics. For example we make extensive use of the second quantification formalism in Section II. We also use the classical BBGKY hierarchy for quantum particles in Section III. The arguments used in these two Sections are also used in the theory of weak turbulence and in general in the derivation of kinetic equations for weakly interacting waves (cf. [4], [27]). Part of the arguments of Section IV are reminiscent of similar ones in [8] and [28]. The main result of the paper is contained in Sections V and VI. We have however included the results in the previous Sections in order to fix the notation and explain the limit under consideration.

II. THE N-PARTICLES SYSTEM AND THE SECOND QUANTIFICATION FORMALISM .

We start recalling the classical derivation of the U-U equation that takes as starting point the study of the dynamics of a quantum N -particle system by means of the second quantification formalism. This will allow us to precise the assumptions in which such a derivation is based.

Let us assume that we have N quantum particles contained in a box $\Omega \equiv [0, L]^3$. We will

denote the density of particles as $n = \frac{N}{L^3}$. We will suppose also that the particles interact by means of pair potentials. The hamiltonian of the system is then given by:

$$H_N = H_{0,N} + H_{1,N} \quad (4)$$

where:

$$H_{0,N} = -\frac{\hbar^2}{2m} \sum_{j=1}^N \Delta_{x_j}, \quad H_{1,N} = \frac{1}{2} \sum_{k=1}^N \sum_{j=1, k \neq j}^N V(x_k - x_j). \quad (5)$$

The evolution of the system is given by means of Heisenberg's equation for the density matrix:

$$i\hbar \partial_t \rho = [H_N, \rho] \quad (6)$$

$$\rho(0) = \rho_0. \quad (7)$$

The precise form of the initial density matrix that characterizes the initial state of the system will be given later. Let us precise the order of magnitude of the several parameters arising in the system. There are two main characteristic length scales in the problem, namely the average distance between particles $d = \frac{L}{N^{1/3}}$, and the De Broglie length that is given by $\lambda = \frac{\hbar}{p_0} \equiv \frac{\hbar}{\sqrt{2mk_B T}}$. By assumption $p_0 \equiv \sqrt{2mk_B T}$ is just an estimate of a typical particle momentum. The temperature T is not a true thermodynamic temperature, because the system is not in equilibrium, but it is just a measure of the characteristic energy for the gas particles.

On the other hand, Born's approximation means in mathematical terms that we may assume that the interaction potential between particles V is smooth, but after deriving a set of kinetic equations we will take the limit $V \rightarrow g \delta(x)$, where g is defined just after the formula (3).

The main assumptions on the physical parameters that we use in this paper are the following:

$$N \gg 1, \quad L \gg 1, \quad n = \frac{N}{L^3} = \text{constant} \quad (8)$$

$$\lambda \sim d \quad (9)$$

$$\frac{m\lambda^2 g}{d^3} \sim \frac{\lambda^2 a}{d^3} \ll 1. \quad (10)$$

Assumption (8) is the usual Thermodynamic limit assumption that ensures that there are not boundary effects on the resulting equations. Assumption (10) means that the particle

interactions are weak and allows us to derive a kinetic equation for the evolution of the distribution of particles. Finally assumption (9) means that the particle densities are large enough to allow for the formation of B-E condensate. This is related to the fact that the kinetic equation obtained under the assumption (9) can yield blow up in finite time

We will first obtain a set of kinetic equations describing the evolution of the solutions of (6), (7) in the limit defined by (8)-(10). This problem was solved by Nordheim and Uehling & Uhlenbeck under the implicit additional hypothesis of the boundedness for the solution of the resulting kinetic equation. However, such assumption fails, because the solution of the limit equation blows up in finite time as it has been seen in the numerical simulations of [18], [22, 23]. Therefore the Uehling-Uhlenbeck equation is not the correct limit for the system of particles under consideration in the limit (8)-(10), if the time t is close to the time of formation of a singularity. The main goal of this paper is to obtain a new kinetic equation describing the distribution of particles during the formation of the condensate.

A. Second quantification formalism.

In order to study the N particle system in the limit (8)-(10) we will use the formalism of the second quantification. Most of the computations in this Subsection are standard, but we will reproduce them for the reader's convenience. We will assume that the hamiltonian H_N in (4), (5) acts in the Hilbert space:

$$\mathcal{H}_N \equiv \bigotimes_{n=0}^N (L^2(\Omega))^n \quad (11)$$

By definiteness we will assume that the wave functions satisfy periodic boundary conditions in Ω . Homogeneous Dirichlet boundary conditions would work similarly. For periodic boundary conditions the eigenvalues of the momentum operator for a single particle $p_k \equiv -i\hbar\partial_{x_k}$ are given by:

$$p = \frac{2\pi\hbar}{L}\ell, \quad \ell \in \mathbb{Z}^3$$

We will denote the normalized eigenfunctions associated to the operator $H_{0,N}$ as:

$$|..., n_\ell, ... \rangle \quad , \quad \ell \in \mathbb{Z}^3$$

where n_ℓ is the number of particles in the state ℓ .

For notational convenience we will use also the following alternative way of writing these eigenfunctions:

$$|..., n_\ell, ... \rangle = |n(\ell)\rangle, \text{ where } n : \mathbb{Z}^3 \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$$

This notation will be convenient to write in a short manner all the possible choices of occupation numbers.

We introduce the well known annihilation and creation operators a_ℓ , a_ℓ^+ , N_ℓ whose action on these eigenfunctions is given by:

$$a_\ell |..., n_\ell, ... \rangle = \sqrt{n_\ell} |..., n_\ell - 1, ... \rangle, \quad \ell \in \mathbb{Z}^3 \quad (12)$$

$$a_\ell^+ |..., n_\ell, ... \rangle = \sqrt{n_\ell + 1} |..., n_\ell + 1, ... \rangle, \quad \ell \in \mathbb{Z}^3 \quad (13)$$

$$N_\ell |..., n_\ell, ... \rangle \equiv a_\ell^+ a_\ell |..., n_\ell, ... \rangle = n_\ell |..., n_\ell, ... \rangle \quad (14)$$

These operators satisfy the commutation relations:

$$[a_k, a_\ell^+] = \delta_{k,\ell}, \quad [a_k, a_\ell] = [a_k^+, a_\ell^+] = 0. \quad (15)$$

We now define the annihilation and creation operators of a particle at the point x of Ω by means of:

$$\psi(x) = \frac{1}{L^{\frac{3}{2}}} \sum_{\ell \in \mathbb{Z}^3} a_\ell e^{\frac{2\pi i \ell x}{L}}, \quad \psi^+(x) = \frac{1}{L^{\frac{3}{2}}} \sum_{\ell \in \mathbb{Z}^3} a_\ell^+ e^{-\frac{2\pi i \ell x}{L}} \quad (16)$$

Notice that using (15):

$$[\psi(x), \psi^+(y)] = \frac{1}{L^3} \sum_{\ell \in \mathbb{Z}^3} e^{\frac{2\pi i \ell (x-y)}{L}} = \delta(x-y)$$

$$[\psi(x), \psi(y)] = [\psi^+(x), \psi^+(y)] = 0$$

Using all these operators we can rewrite the operator $H_{0,N}$ as:

$$H_{0,N} = \sum_{j=1}^N \frac{p_j^2}{2m} = \sum_{\ell \in \mathbb{Z}^3} \epsilon_\ell a_\ell^+ a_\ell$$

where:

$$\epsilon_\ell \equiv \frac{4\pi^2 \hbar^2 \ell^2}{2mL^2}, \quad \ell \in \mathbb{Z}^3$$

Taking the gradient of (16) we obtain: whence:

$$H_{0,N} = \frac{\hbar^2}{2m} \int_{\Omega} \nabla \psi^+(x) \nabla \psi(x) dx \quad (17)$$

On the other hand:

$$H_{1,N} = \frac{1}{2} \int_{\Omega} dx_1 \int_{\Omega} dx_2 V(x_1 - x_2) \psi^+(x_1) \psi^+(x_2) \psi(x_1) \psi(x_2) \quad (18)$$

We define the distribution functions :

$$f_{j,m}(x_1, \dots, x_j; y_1, \dots, y_m) \equiv \text{Tr}(\rho \psi^+(y_1) \psi^+(y_2) \dots \psi^+(y_m) \psi(x_1) \psi(x_2) \dots \psi(x_j)) \quad (19)$$

The computation of the evolution equations for the functions $f_{j,m}$ is standard (see for example [1]) . Using (6) it follows that:

$$i\hbar \partial_t f_{j,m} = \text{Tr}(\rho [\psi^+(y_1) \psi^+(y_2) \dots \psi^+(y_m) \psi(x_1) \psi(x_2) \dots \psi(x_j), H_N]) \quad (20)$$

On the other hand, we can compute the commutator in (20) to obtain the following evolution equation for the distribution functions $f_{j,m}$:

$$\begin{aligned} i\hbar \partial_t f_{j,m}(x_1, \dots, x_j; y_1, \dots, y_m) = & -\frac{\hbar^2}{2m} \left(\sum_{s=1}^j \Delta_{x_s} - \sum_{s=1}^m \Delta_{y_s} \right) f_{j,m}(x_1, \dots, x_j; y_1, \dots, y_m) + \\ & + \int_{\Omega} d\xi \left[\sum_{k=1}^j V(\xi - x_k) - \sum_{k=1}^m V(\xi - y_k) \right] f_{j+1,m+1}(x_1, \dots, x_j, \xi; y_1, \dots, y_m, \xi) + \\ & + \frac{1}{2} \left[\sum_{k=1}^j \sum_{s=1, k \neq s}^j V(x_k - x_s) - \sum_{k=1}^m \sum_{s=1}^m V(y_k - y_s) \right] f_{j,m}(x_1, \dots, x_j; y_1, \dots, y_m). \end{aligned} \quad (21)$$

B. On the choice of the initial data.

In order to solve the system of equations (21) we must prescribe suitable initial data. We will assume that the initial matrix density $\rho(0)$ satisfies:

$$\rho(0) = \rho_0 = \frac{1}{Q} \sum_{n: \mathbb{Z}^3 \rightarrow N} P_0(z, \Theta; n) |n\rangle \langle n| \quad (22)$$

$$P_0(z, \Theta; n) \equiv z^{[\sum_{\ell \in \mathbb{Z}^3} n(\ell)]} \left(\prod_{\ell \in \mathbb{Z}^3} (\theta_{\ell})^{n(\ell)} \right) \quad (23)$$

$$n: \mathbb{Z}^3 \rightarrow \mathbb{N}, \quad \sum_{\ell \in \mathbb{Z}^3} n(\ell) < \infty \quad (24)$$

where:

$$Q = \sum_{n: \mathbb{Z}^3 \rightarrow N} P_0(z, \Theta; n)$$

has been chosen in order to have $\text{Tr}(\rho_0) = 1$, and where:

$$\theta_\ell \equiv \Theta \left(\frac{2\pi^2 \hbar^2}{mk_B T} \frac{\ell^2}{L^2} \right).$$

Choosing the initial data as in (22), (23) we are assuming that the particles are independently and homogeneously distributed in space according to the distribution $\Theta(\cdot)$ in the space of energy. Since we use a macrocanonical distribution the number of variables is a stochastic variable. In the thermodynamic limit the fluctuations in the number of particles can be expected to disappear as it is usual in statistical physics. The value of z is chosen to obtain a given average number of particles N for the distribution. Therefore:

$$\langle N \rangle = \text{Tr}(\rho_0 N) = \frac{1}{Q} \sum_{n: \mathbb{Z}^3 \rightarrow N} P_0(z, \Theta; n) N(n) = z \frac{\partial (\log(Q))}{\partial z}$$

where $N(n) = \sum_{\ell \in \mathbb{Z}^3} n(\ell)$.

Instead of analyzing the original system (6), (7), we will study the equivalent system of equations (21) that is more convenient to use perturbative arguments. Due to (19) we must solve these equations with initial data:

$$\begin{aligned} f_{j,m;0}(x_1, \dots, x_j; y_1, \dots, y_m) &\equiv \\ \text{Tr}(\rho_0 \psi^+(x_1) \psi^+(x_2) \dots \psi^+(x_j) \psi(y_1) \psi(y_2) \dots \psi(y_m)) &. \end{aligned} \quad (25)$$

Using (22), (23) we obtain:

$$f_{j,m;0}(x_1, \dots, x_j; y_1, \dots, y_m) = 0 \quad \text{if } j \neq m \quad (26)$$

The evolution equations (21), with initial data (26) admit a solution satisfying:

$$f_{j,m}(x_1, \dots, x_j; y_1, \dots, y_m; t) = 0 \quad \text{if } j \neq m$$

Therefore, we can restrict our study to the functions:

$$F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) \equiv f_{k,k}(x_1, \dots, x_k; y_1, \dots, y_k; t)$$

On the other hand we can compute the initial distribution

$F_{k,0}(x_1, \dots, x_k; y_1, \dots, y_k) \equiv F_k(x_1, \dots, x_k; y_1, \dots, y_k; 0)$ that due to (16) and (25) is given by:

$$F_{k,0}(x_1, \dots, x_k; y_1, \dots, y_k) = \frac{1}{L^{3k}} \sum_{\ell_1 \in \mathbb{Z}^3} \dots \sum_{\ell_k \in \mathbb{Z}^3} \sum_{j_1 \in \mathbb{Z}^3} \dots \sum_{j_k \in \mathbb{Z}^3} e^{-\frac{2\pi i(\ell_1 x_1 + \dots + \ell_k x_k)}{L} + \frac{2\pi i(j_1 y_1 + \dots + j_k y_k)}{L}} \times \\ \times \text{Tr}(\rho_0 a_{\ell_1}^+ a_{\ell_2}^+ \dots a_{\ell_k}^+ a_{j_1} \dots a_{j_k}). \quad (27)$$

By (22) we then have:

$$\text{Tr}(\rho_0 a_{\ell_1}^+ a_{\ell_2}^+ \dots a_{\ell_k}^+ a_{j_1} \dots a_{j_k}) = \frac{1}{Q} \sum_{n: \mathbb{Z}^3 \rightarrow N} \prod_{\ell \in \mathbb{Z}^3} (z\theta_\ell)^{n(\ell)} \langle n | a_{\ell_1}^+ a_{\ell_2}^+ \dots a_{\ell_k}^+ a_{j_1} \dots a_{j_k} | n \rangle$$

We now compute the terms $\langle n | a_{\ell_1}^+ a_{\ell_2}^+ \dots a_{\ell_k}^+ a_{j_1} \dots a_{j_k} | n \rangle$. We will assume for the moment that all the terms j_1, \dots, j_k are different. In this case, the matrix element $\langle n | a_{\ell_1}^+ a_{\ell_2}^+ \dots a_{\ell_k}^+ a_{j_1} \dots a_{j_k} | n \rangle$ is not zero if and only if the coefficients ℓ_1, \dots, ℓ_k are the same as the j_1, \dots, j_k or a permutation of them. Therefore:

$$\langle n | a_{\ell_1}^+ a_{\ell_2}^+ \dots a_{\ell_k}^+ a_{j_1} \dots a_{j_k} | n \rangle = \left[\prod_{s=1}^k n(j_s) \right] \sum_{\sigma \in S^k} \delta_{j_{\sigma(1)}, \ell_1} \delta_{j_{\sigma(2)}, \ell_2} \dots \delta_{j_{\sigma(k)}, \ell_k} \quad (28)$$

where S^k is the group of permutations of the elements $\{1, \dots, k\}$. If some of the coefficients j_1, \dots, j_k are repeated, the scalar product on the left hand side of (28) can be bounded, using standard statistical physics computations (cf. [3]), as $\frac{C}{L^3} \left[\prod_{s=1}^k n(j_s) \right]$. Therefore, using formula (28) again and replacing some sums by Riemann integrals, we can approximate (27) in the limit $L \rightarrow \infty$ as

$$F_{k,0}(x_1, \dots, x_k; y_1, \dots, y_k) = \sum_{\sigma \in S^k} \prod_{m=1}^k F_0(y_m - x_{\sigma(m)}; z) \quad (29)$$

where:

$$F_0(y; z) \equiv \int_{\mathbb{R}^3} \left[\frac{z\Theta(\bar{\epsilon}(\xi)) e^{2\pi i y \xi}}{1 - z\Theta(\bar{\epsilon}(\xi))} \right] d\xi, \quad \bar{\epsilon}(\xi) \equiv \frac{2\pi^2 \hbar^2}{mk_B T} \xi^2. \quad (30)$$

III. THE SMALL CORRELATIONS APPROXIMATION.

A. Non dimensional equations.

Summarizing, we have reduced the problem to the following system of equations (cf. (21), (29)):

$$i\hbar\partial_t F_k(x_1, \dots, x_k; y_1, \dots, y_k, t) = A_1 + A_2 + A_3; \quad k = 1, 2, \dots \quad (31)$$

$$A_1 = -\frac{\hbar^2}{2m} \left(\sum_{s=1}^k [\Delta_{x_s} - \Delta_{y_s}] \right) F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) \quad (32)$$

$$A_2 = \int_{\Omega} d\xi \left[\sum_{j=1}^k [V(\xi - x_j) - V(\xi - y_j)] \right] F_{k+1}(x_1, \dots, x_k, \xi; y_1, \dots, y_k, \xi; t) \quad (33)$$

$$A_3 = \frac{1}{2} \left[\sum_{j=1}^k \sum_{s=1, j \neq s}^k [V(x_j - x_s) - V(y_j - y_s)] \right] F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) \quad (34)$$

with initial data:

$$F_{k,0}(x_1, \dots, x_k; y_1, \dots, y_k) = \sum_{\sigma \in S^k} \prod_{m=1}^k F_0(y_m - x_{\sigma(m)}; z). \quad (35)$$

The two first equations of this hierarchy are:

$$\begin{aligned} i\hbar\partial_t F_1(x_1; y_1; t) &= -\frac{\hbar^2}{2m} (\Delta_{x_1} - \Delta_{y_1}) F_1(x_1; y_1; t) \\ &+ \int_{\Omega} d\xi [V(\xi - x_1) - V(\xi - y_1)] [F_1(x_1; y_1; t) F_1(\xi; \xi; t) + F_1(x_1; \xi; t) F_1(\xi; y_1; t)] \\ &+ \int_{\Omega} d\xi [V(\xi - x_1) - V(\xi - y_1)] F_2(x_1, \xi; y_1, \xi; t) \end{aligned} \quad (36)$$

$$\begin{aligned} i\hbar\partial_t F_2(x_1, x_2; y_1, y_2; t) &= -\frac{\hbar^2}{2m} (\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}) F_2(x_1, x_2; y_1, y_2; t) + \\ &+ \sum_{j=1}^2 \int_{\Omega} d\xi [V(\xi - x_j) - V(\xi - y_j)] F_3(x_1, x_2, \xi; y_1, y_2, \xi; t) + \\ &+ [V(x_1 - x_2) - V(y_1 - y_2)] F_2(x_1, x_2; y_1, y_2; t) \end{aligned} \quad (37)$$

In order to understand more clearly the limit that we are considering, we introduce the non-dimensional variables:

$$x = \lambda \hat{x}, \quad V(x) = \frac{g}{\lambda^3} \hat{V}(\hat{x}), \quad F_k(x) = \left(\frac{1}{d^3} \right)^k \hat{F}_k(\hat{x}), \quad t = \frac{2m\lambda^2}{\hbar\epsilon^2} \hat{t}, \quad p = \frac{\hbar}{\lambda} \hat{p} \quad (38)$$

$$g = \epsilon \frac{\hbar^2}{2m\lambda^2} d^3 \quad (39)$$

where, due to (10), ϵ is a small parameter and by assumption, the potential $\widehat{V}(\hat{x})$ is now of order one. Our choice of time scale is due to the fact that we want to obtain, in the limit $\epsilon \rightarrow 0$, an equation in which the particle density varies in times \hat{t} of order one. Notice that $(x, p) \rightarrow (\hat{x}, \hat{p})$ is not a canonical transformation but transforms a “quantum cell” in the phase space of volume \hbar into another cell of volume one.

For the sake of simplicity we drop the hats in all the variables \hat{x} , \hat{p} , \hat{t} , \widehat{F}_k and \widehat{V} as it is customary in the computations of asymptotic expansions. The system (36)-(37) then becomes

$$\begin{aligned} i\partial_t F_1(x_1; y_1; t) = & -\frac{1}{\epsilon^2} (\Delta_{x_1} - \Delta_{y_1}) F_1(x_1; y_1; t) \\ & + \frac{1}{\epsilon} \int_{\Omega} d\xi [V(\xi - x_1) - V(\xi - y_1)] [F_1(x_1; y_1; t) F_1(\xi; \xi; t) + F_1(x_1; \xi; t) F_1(\xi; y_1; t)] \\ & + \frac{1}{\epsilon} \int_{\Omega} d\xi [V(\xi - x_1) - V(\xi - y_1)] F_2(x_1, \xi; y_1, \xi; t) \end{aligned} \quad (40)$$

$$\begin{aligned} i\partial_t F_2(x_1, x_2; y_1, y_2; t) = & -\frac{1}{\epsilon^2} (\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}) F_2(x_1, x_2; y_1, y_2; t) + \\ & + \frac{1}{\epsilon} \sum_{j=1}^2 \int_{\Omega} d\xi [V(\xi - x_j) - V(\xi - y_j)] F_3(x_1, x_2, \xi; y_1, y_2, \xi; t) + \\ & + \frac{1}{\epsilon} \left(\frac{d}{\lambda} \right)^3 [V(x_1 - x_2) - V(y_1 - y_2)] F_2(x_1, x_2; y_1, y_2; t) \end{aligned} \quad (41)$$

B. Small correlations limit.

Our aim is to obtain closure relations for the functions F_k by means of a perturbative argument.

Notice that in the absence of potential the system of equations (40), (41) might be explicitly solved and the resulting solutions have the form:

$$F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) = \sum_{\sigma \in S^k} \prod_{m=1}^k F_1(x_{\sigma(m)}, y_m; t) \quad (42)$$

Moreover, the function F_0 in (30) is invariant under spatial translations whence $F_0(x_1; y_1) = F_0(x_1 - y_1)$. Since the system of equations (31)-(34) are also invariant under spatial translations it follows that $F_1(x_1; y_1; t) = F_1(x_1 - y_1; t)$ for any $t > 0$.

Notice that in this case we can think in the solutions of this form as “uncorrelated” solutions, although in a strict mathematical sense the corresponding probability distributions are not uncorrelated, but the only correlations between particles whose distribution is given by (42) would be the ones due to the symmetry of the wave functions due to the bosonic character of the particles (cf. the discussion in [3]). In any case the approximation (42) is a convenient starting point for the computation of the solutions of (31)-(35) in a perturbative manner. We define the correlation functions G_k by means of the identity:

$$F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) = G_k(x_1, \dots, x_k; y_1, \dots, y_k; t) + \tilde{F}_k(x_1, \dots, x_k; y_1, \dots, y_k; t). \quad (43)$$

where we have defined

$$\tilde{F}_k(x_1, \dots, x_k; y_1, \dots, y_k; t) = \sum_{\sigma \in S^k} \prod_{m=1}^k F_1(x_{\sigma(m)}, y_m; t). \quad (44)$$

It is possible to derive a kinetic approximation for (36) (37) under the following small correlation assumptions

$$|G_k| << \prod_{m=1}^k |F_1|. \quad (45)$$

Indeed, under this assumption we obtain, plugging (43) into (36)-(37):

$$\begin{aligned} i\partial_t F_1(x_1; y_1; t) &= -\frac{1}{\epsilon^2} (\Delta_{x_1} - \Delta_{y_1}) F_1(x_1; y_1; t) \\ &+ \frac{1}{\epsilon} \int_{\Omega} d\xi [V(\xi - x_1) - V(\xi - y_1)] [F_1(x_1; y_1; t) F_1(\xi; \xi; t) + F_1(x_1; \xi; t) F_1(\xi; y_1; t)] \\ &+ \frac{1}{\epsilon} \int_{\Omega} d\xi [V(\xi - x_1) - V(\xi - y_1)] F_2(x_1, \xi; y_1, \xi; t) \end{aligned} \quad (46)$$

$$\begin{aligned} i\partial_t F_2(x_1, x_2; y_1, y_2; t) &= -\frac{1}{\epsilon^2} (\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}) F_2(x_1, x_2; y_1, y_2; t) + \\ &+ \frac{1}{\epsilon} \sum_{j=1}^2 \int_{\Omega} d\xi [V(\xi - x_j) - V(\xi - y_j)] \tilde{F}_3(x_1, x_2, \xi; y_1, y_2, \xi; t) + \\ &+ \frac{1}{\epsilon} \left(\frac{d}{\lambda} \right)^3 [V(x_1 - x_2) - V(y_1 - y_2)] \tilde{F}_2(x_1, x_2; y_1, y_2; t) \end{aligned} \quad (47)$$

The relative strength of the terms yielding correlations is of order ϵ . This explains why in equation (47) we have approximated F_2 and F_3 by \tilde{F}_2 and \tilde{F}_3 respectively. Notice that we have kept all the terms in the equation (46) and only terms of order $1/\epsilon$ or larger in (47).

We now compute the evolution equation for $G_2(x_1, x_2; y_1, y_2; t)$ using (43) and the approximation (46)-(47):

$$i\partial_t G_2(x_1, x_2; y_1, y_2; t) = -\frac{1}{\epsilon^2} (\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}) G_2(x_1, x_2; y_1, y_2; t) + \frac{1}{\epsilon} \left(\frac{d}{\lambda}\right)^3 [V(x_1 - x_2) - V(y_1 - y_2)] \tilde{F}_2(x_1, x_2; y_1, y_2; t) + \frac{1}{\epsilon} \int_{\Omega} d\xi H(\xi; t) \quad (48)$$

$$H(\xi; t) = \sum_{j=1}^2 [V(\xi - x_j) - V(\xi - y_j)] \tilde{F}_3(x_1, x_2, \xi; y_1, y_2, \xi; t) - \sum_{\sigma \in S^2} [V(\xi - x_1) - V(\xi - y_{\sigma(1)})] F_2(x_1, \xi; y_{\sigma(1)}, \xi; t) F_1(x_2; y_{\sigma(2)}; t) - \sum_{\sigma \in S^2} F_1(x_1; y_{\sigma(1)}; t) [V(\xi - x_2) - V(\xi - y_{\sigma(2)})] F_2(x_2, \xi; y_{\sigma(2)}, \xi; t) \quad (49)$$

After some computations, neglecting terms of order $\mathcal{O}(\epsilon^2)$ it follows that:

$$H(\xi; t) = V(\xi - x_1) F_1(x_2; \xi; t) \tilde{F}_2(x_1, \xi; y_1, y_2; t) - V(\xi - y_1) F_1(\xi; y_2; t) \tilde{F}_2(x_1, x_2; y_1, \xi; t) + V(\xi - x_2) F_1(x_1; \xi; t) \tilde{F}_2(x_2, \xi; y_1, y_2; t) - V(\xi - y_2) F_1(\xi; y_1; t) \tilde{F}_2(x_1, x_2; y_2, \xi; t). \quad (50)$$

Plugging (50) into (48) we obtain:

$$i\partial_t G_2(x_1, x_2; y_1, y_2; t) = -\frac{1}{\epsilon^2} (\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}) G_2(x_1, x_2; y_1, y_2; t) + \frac{1}{\epsilon} \int \left[V(\xi - x_1) F_1(x_2; \xi; t) \tilde{F}_2(x_1, \xi; y_1, y_2; t) - V(\xi - y_1) F_1(\xi; y_2; t) \tilde{F}_2(x_1, x_2; y_1, \xi; t) \right] d\xi + \frac{1}{\epsilon} \int \left[V(\xi - x_2) F_1(x_1; \xi; t) \tilde{F}_2(x_2, \xi; y_1, y_2; t) - V(\xi - y_2) F_1(\xi; y_1; t) \tilde{F}_2(x_1, x_2; y_2, \xi; t) \right] d\xi + \frac{1}{\epsilon} \left(\frac{d}{\lambda}\right)^3 [V(x_1 - x_2) - V(y_1 - y_2)] \tilde{F}_2(x_1, x_2; y_1, y_2; t), \quad (51)$$

where, due to (9), d/λ is of order one.

The equations (46) and (51) provide the evolution equations for the functions F_1 and G_2 . We have not used yet Born's approximation which in the variables that we are using reduces to

$$V(x) = \delta(x) \quad (52)$$

Using (52) we obtain:

$$i\partial_t F_1(x_1; y_1; t) = -\frac{1}{\epsilon^2} (\Delta_{x_1} - \Delta_{y_1}) F_1(x_1; y_1; t) + \frac{1}{\epsilon} g [G_2(x_1, x_1; y_1, x_1; t) - G_2(x_1, y_1; y_1, y_1; t)] \quad (53)$$

$$\begin{aligned} i\partial_t G_2(x_1, x_2; y_1, y_2; t) = & -\frac{1}{\epsilon^2} (\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}) G_2(x_1, x_2; y_1, y_2; t) \\ & + \frac{1}{\epsilon} g \left[F_1(x_2; x_1; t) \tilde{F}_2(x_1, x_1; y_1, y_2; t) - F_1(y_1; y_2; t) \tilde{F}_2(x_1, x_2; y_1, y_1; t) \right] + \\ & + \frac{1}{\epsilon} g \left[F_1(x_1; x_2; t) \tilde{F}_2(x_2, x_2; y_1, y_2; t) - F_1(y_2; y_1; t) \tilde{F}_2(x_1, x_2; y_2, y_2; t) \right] \\ & + \frac{1}{\epsilon} \left(\frac{d}{\lambda} \right)^3 g [\delta(x_1 - x_2) - \delta(y_1 - y_2)] \tilde{F}_2(x_1, x_2; y_2, y_2; t) \end{aligned} \quad (54)$$

The invariance of the initial distribution $F_0(x_1; y_1)$ under spatial translations imply that, with a slight abuse of language, the solutions of (53), (54) have the form:

$$F_1(x_1; y_1; t) = F_1(x_1 - y_1; t) \quad (55)$$

$$G_2(x_1, x_2; y_1, y_2; t) = G_2(x_1 - y_1, x_2 - y_1; 0, y_2 - y_1; t) \quad (56)$$

Under these assumptions the equations (53), (54) reduce to:

$$i\partial_t F_1(x_1 - y_1; t) = \frac{1}{\epsilon} g [G_2(x_1, x_1; y_1, x_1; t) - G_2(x_1, y_1; y_1, y_1; t)] \quad (57)$$

$$i\partial_t G_2(x_1, x_2; y_1, y_2; t) = -\frac{1}{\epsilon^2} (\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}) G_2(x_1, x_2; y_1, y_2; t) + \frac{1}{\epsilon} Q[F_1] \quad (58)$$

where:

$$\begin{aligned} Q[F_1](x_1, x_2; y_1, y_2; t) = & \left[F_1(x_2; x_1; t) \tilde{F}_2(x_1, x_1; y_1, y_2; t) - F_1(y_1; y_2; t) \tilde{F}_2(x_1, x_2; y_1, y_1; t) \right] \\ & + \left[F_1(x_1; x_2; t) \tilde{F}_2(x_2, x_2; y_1, y_2; t) - F_1(y_2; y_1; t) \tilde{F}_2(x_1, x_2; y_2, y_2; t) \right] \\ & + \left(\frac{d}{\lambda} \right)^3 [\delta(x_1 - x_2) - \delta(y_1 - y_2)] \tilde{F}_2(x_1, x_2; y_2, y_2; t) \end{aligned} \quad (59)$$

Due to (35) we have:

$$G_2(x_1, x_2; y_1, y_2; 0) \equiv 0. \quad (60)$$

The system of equations (57)-(60) will be our starting point for the description of the Bose gas in which we are interested. Notice that it is a closed system of partial differential equations.

IV. THE PROBLEM IN THE PHASE SPACE.

The function that describes the one-particle density in the phase space in quantum problems is the Wigner transform of $F_1(x_1; y_1; t)$. Such a function is defined as:

$$f_1(x, p; t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} F_1(x + \zeta; x - \zeta; t) e^{i\zeta p} d\zeta \quad (61)$$

where the normalization constant in (61) has been chosen in order to have

$$\int f_1(x, p) dx dp = N.$$

In the spatially homogeneous case we have $F_1(x + \zeta; x - \zeta; t) = F_1(\zeta; t)$ due to (56). Therefore, (61) reduces to the Fourier transform:

$$f_1(x, p; t) = f_1(p, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} F_1(\zeta; t) e^{i\zeta p} d\zeta \quad (62)$$

In order to obtain the evolution equation for $f_1(p, t)$ we then take the Fourier transform of (58), (59):

$$i\partial_t f_1(p, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} [G_2(\zeta, \zeta; 0, \zeta; t) - G_2(\zeta, 0; 0, 0; t)] e^{i\zeta p} d\zeta \quad (63)$$

On the other hand we have the following Fourier representation for the function $G_2(x_1, x_2; y_1, y_2; t)$:

$$g_2(\xi_1, \xi_2; \eta_1, \eta_2; t) = \frac{1}{(2\pi)^{12}} \int_{(\mathbb{R}^3)^4} dx_1 dx_2 dy_1 dy_2 e^{i(\xi_1 x_1 + \xi_2 x_2 - \eta_1 y_1 - \eta_2 y_2)} G_2(x_1, x_2; y_1, y_2; t) \quad (64)$$

$$G_2(x_1, x_2; y_1, y_2; t) = \int_{(\mathbb{R}^3)^4} d\xi_1 d\xi_2 d\eta_1 d\eta_2 e^{-i(\xi_1 x_1 + \xi_2 x_2 - \eta_1 y_1 - \eta_2 y_2)} g_2(\xi_1, \xi_2; \eta_1, \eta_2; t). \quad (65)$$

Let us write:

$$w(\xi_1, \xi_2; \eta_1, \eta_2; t) = \frac{1}{(2\pi)^{12}} \int_{(\mathbb{R}^3)^4} dx_1 dx_2 dy_1 dy_2 e^{i(\xi_1 x_1 + \xi_2 x_2 - \eta_1 y_1 - \eta_2 y_2)} Q[F_1](x_1, x_2; y_1, y_2; t) \quad (66)$$

Taking the Fourier transform of (58) and using (65) in (63) we obtain the following system of equations for f_1 and g_2 :

$$i\partial_t f_1(p, t) = \frac{1}{\epsilon} \int_{(\mathbb{R}^3)^4} d\xi_1 d\xi_2 d\eta_1 d\eta_2 [\delta(p - \eta_1) - \delta(p - \xi_1)] g_2(\xi_1, \xi_2; \eta_1, \eta_2; t) \quad (67)$$

$$i\partial_t g_2(\xi_1, \xi_2; \eta_1, \eta_2; t) = \frac{1}{\epsilon^2} [\varepsilon(\xi_1) + \varepsilon(\xi_2) - \varepsilon(\eta_1) - \varepsilon(\eta_2)] g_2(\xi_1, \xi_2; \eta_1, \eta_2; t) + \frac{1}{\epsilon} w(\xi_1, \xi_2; \eta_1, \eta_2; t) \quad (68)$$

where $w(\xi_1, \xi_2; \eta_1, \eta_2; t)$ is as in (59), (66) and the energy $\varepsilon(p)$ in the non-dimensional variables is $\varepsilon(p) = p^2$. To obtain a closed system for f_1, g_2 it only remains to compute $w(\xi_1, \xi_2; \eta_1, \eta_2; t)$ in terms of f_1 . To this end notice that (62) yields:

$$F_1(\zeta; t) = \int f_1(p, t) e^{-ip\zeta} dp \quad (69)$$

Using (55) in the formula of $Q[F_1]$ and plugging the final expression in (66) we obtain, after some computations:

$$\begin{aligned} w(\xi_1, \xi_2; \eta_1, \eta_2; t) &= 2\delta(\xi_1 + \xi_2 - \eta_1 - \eta_2) q[f](\xi_1, \xi_2; \eta_1, \eta_2, t) \\ q[f_1](\xi_1, \xi_2; \eta_1, \eta_2, t) &= \left[f_1(\eta_1) f_1(\eta_2) \left(\left(\frac{d}{2\pi\lambda} \right)^3 + f_1(\xi_1) \right) \left(\left(\frac{d}{2\pi\lambda} \right)^3 + f_1(\xi_2) \right) - \right. \\ &\quad \left. - f_1(\xi_1) f_1(\xi_2) \left(\left(\frac{d}{2\pi\lambda} \right)^3 + f_1(\eta_1) \right) \left(\left(\frac{d}{2\pi\lambda} \right)^3 + f_1(\eta_2) \right) \right]. \end{aligned} \quad (70)$$

where we have dropped the time dependence of the function f_1 in the right hand side of (70). The solution g_2 to (68) is then:

$$\begin{aligned} g_2(\xi_1, \xi_2; \eta_1, \eta_2; t) &= -\frac{2i}{\epsilon} \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2) \int_0^t e^{-\frac{i}{\epsilon^2} [\varepsilon(\xi_1) + \varepsilon(\xi_2) - \varepsilon(\eta_1) - \varepsilon(\eta_2)](t-s)} \times \\ &\quad \times q[f_1](\xi_1, \xi_2; \eta_1, \eta_2; s) ds. \end{aligned} \quad (71)$$

where we have used that $g_2(\cdot, \cdot; \cdot, \cdot; 0) \equiv 0$ due to (60) and (64). The Dirac measure in (71) may be simplified if we define,

$$g_2(\xi_1, \xi_2; \eta_1, \eta_2; t) = \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2) \varphi(\xi_1, \xi_2; \eta_1, \eta_2; t),$$

from where, (71) gives

$$\varphi(\xi_1, \xi_2; \eta_1, \eta_2; t) = -\frac{2i}{\epsilon} \int_0^t e^{-\frac{i}{\epsilon^2} [\varepsilon(\xi_1) + \varepsilon(\xi_2) - \varepsilon(\eta_1) - \varepsilon(\eta_2)](t-s)} q[f_1](\xi_1, \xi_2; \eta_1, \eta_2; s) ds \quad (72)$$

Plugging (72) in (67) and using the symmetry of $q[f_1]$ with respect to its arguments ξ_1, ξ_2, η_1 and η_2 we finally obtain the following equation for f_1 :

$$\partial_t f_1(p_1, t) = \frac{4}{\epsilon^2} \int_0^t ds \int_{(\mathbb{R}^3)^3} dp_2 dp_3 dp_4 \left\{ \cos \left[\frac{1}{\epsilon^2} (\varepsilon(p_1) + \varepsilon(p_2) - \varepsilon(p_3) - \varepsilon(p_4))(t - s) \right] \right\} \\ \times \delta(p_1 + p_2 - p_3 - p_4) q[f_1](p_1, p_2; p_3, p_4; s) \quad (73)$$

Non-Markovian Boltzmann equations have been found in several physical situations (cf. for example [2], [8], [11], [17], [28] and references therein).

V. THE KINETIC LIMIT: THE UEHLING UHLENBECK EQUATION.

A. Derivation of the Uehling Uhlenbeck equation.

The formal derivation of the U-U equation would then proceed as follows. If we suppose that

$$\frac{1}{\epsilon^2} (\varepsilon(p) + \varepsilon(p_2) - \varepsilon(p_3) - \varepsilon(p_4))(\tau - \sigma) \gg 1 \quad (74)$$

a simple formal argument gives:

$$\frac{1}{\epsilon^2} \cos \left[\frac{1}{\epsilon^2} (\varepsilon(p) + \varepsilon(p_2) - \varepsilon(p_3) - \varepsilon(p_4))(\tau - \sigma) \right] \rightarrow \\ \pi \delta(\tau - \sigma) \delta(\varepsilon(p) + \varepsilon(p_2) - \varepsilon(p_3) - \varepsilon(p_4)) \quad (75)$$

in the sense of measures, where in that formula $p_2 \equiv p_3 + p_4 - p_1$. We finally end up with the U-U equation:

$$\partial_t f_1(p_1, t) = 4\pi \int_{(\mathbb{R}^3)^3} dp_2 dp_3 dp_4 \delta(\varepsilon(p_1) + \varepsilon(p_2) - \varepsilon(p_3) - \varepsilon(p_4)) \\ \times \delta(p_1 + p_2 - p_3 - p_4) q[f_1](p_1, p_2; p_3, p_4; t) \quad (76)$$

Notice however that this approximation requires the condition (74), which, using the original physical variables, can be formulated as the quasiclassical condition $\Delta E \Delta t \gg \hbar$. Equation (76) is just the equation (1)-(3) written in a different system of units.

B. The loss of validity of the kinetic approximation.

According to the blow up scenario of Semikoz & Tkachev (cf. [22, 23]) and Pomeau et al. (cf. [15, 18]) the blow up for the equation (76) takes place in a self similar manner and the distribution of particles has relevant variations in the regions of the space of momentum p whose size rescales like the power $(T - t)^\beta$ for some positive β . In order to describe this region by means of an equation free of parameters we look for self similar solutions of (76). Such solutions have the form

$$f(t, p) = (T - t)^{-2\beta-1/2} \Phi(\xi), \quad \xi = \frac{p}{(T - t)^\beta} \quad (77)$$

where the numerically computed value of β is such that $\beta = 1.069$ (see [18]). The function Φ , that is of order one, solves then an integro differential equation free of parameters. Notice also that the functional form (77) tells us immediately the time scales for which the interference effects in (73) cannot be ignored, or more precisely, in dimensional variables, the time scale where $\Delta E \Delta t \sim \hbar$. This happens if $p^2(T - t) \sim \epsilon^2$ or equivalently if

$$(T - t) \sim \frac{\epsilon^2}{p^2}. \quad (78)$$

Since $p \sim (T - t)^\beta$ in the self similar region, we obtain that the interference effects appear at times

$$(T - t) \sim \epsilon^{\frac{2}{2\beta+1}} \quad (79)$$

For this time scales we have to introduce a boundary layer in order to take the interference effects into account.

C. The correlations become of order one in the boundary layer.

It turns out that in the same time scale where (75) starts failing, the small correlation approximation condition (45) ceases being valid. Indeed, assuming the self similar behaviour (77) we obtain:

$$F_1(\zeta, t) = (2\pi)^3 \int_{\mathbb{R}^3} f_1(p, t) e^{-i\zeta p} dp \quad (80)$$

$$= (2\pi)^3 (T - t)^{\beta-1/2} \int \Phi(Z) e^{-i\zeta (T-t)^\beta Z} dZ \quad (81)$$

$$= (T - t)^{\beta-1/2} \Psi(Z(T - t)^\beta). \quad (82)$$

On the other hand, (58) and (59) yield that for the boundary layer time scale:

$$G \sim \frac{1}{\epsilon} F_1^3 (T - t)$$

from where, we obtain, using (79):

$$G \sim (T - t)^{2\beta-1} \sim F_1^2. \quad (83)$$

A similar argument shows that $|G_k| \sim \Pi_{m=1}^k |F_1|$ for $k > 1$. It then follows that the approximation of the system (36) (37) by system (46) (47) breaks down at the time scale (79).

VI. THE BOUNDARY LAYER: ANALYTIC DESCRIPTION.

In this Section we derive the set of equations describing the boundary layer where the kinetic approximation is lost. Since, as we have seen, the correlations become of order one in that region, we need to keep a major part of the equations in system (31). Using the non-dimensional variables (38), that system becomes

$$i\partial_t F_k(x_1, \dots, x_k; y_1, \dots, y_k, t) = A_1 + A_2 + A_3; \quad k = 1, 2, \dots \quad (84)$$

$$A_1 = -\frac{1}{\epsilon^2} \left(\sum_{s=1}^k [\Delta_{x_s} - \Delta_{y_s}] \right) F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) \quad (85)$$

$$A_2 = \frac{1}{\epsilon} \int_{\Omega} d\xi \left[\sum_{j=1}^k [V(\xi - x_j) - V(\xi - y_j)] \right] F_{k+1}(x_1, \dots, x_k, \xi; y_1, \dots, y_k, \xi; t) \quad (86)$$

$$A_3 = \frac{1}{2\epsilon} \left(\frac{d}{\lambda} \right)^3 \left[\sum_{j=1}^k \sum_{s=1, k \neq s}^k [V(x_j - x_s) - V(y_j - y_s)] \right] \times \\ \times F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) \quad (87)$$

The rescaling (79) suggest to define new variables as follows:

$$F_k(x_1, \dots, x_k; y_1, \dots, y_k; t) = \epsilon^{\frac{2\beta-1}{2\beta+1}k} H_k(X_1, \dots, X_k; Y_1, \dots, Y_k; \tau) \quad (88)$$

$$T - t = -\epsilon^{\frac{2}{2\beta+1}} \tau, \quad x_i = \epsilon^{-\frac{2\beta}{2\beta+1}} X_i, \quad p = \epsilon^{\frac{2\beta}{2\beta+1}} P \quad (89)$$

Neglecting lower order terms in ϵ and using that $V(x) = \delta(x)$ we obtain that the functions H_k satisfy at leading order the following system:

$$\left\{ \begin{array}{l} i\partial_\tau H_k(X_1, \dots, X_k; Y_1, \dots, Y_k, \tau) = A_1 + A_2; \quad k = 1, 2, \dots \\ A_1 = - \left(\sum_{s=1}^k [\Delta_{X_s} - \Delta_{Y_s}] \right) H_k(X_1, \dots, X_k; Y_1, \dots, Y_k; \tau) \\ A_2 = \sum_{j=1}^k [H_{k+1}(X_1, \dots, X_k, X_j; Y_1, \dots, Y_k, X_j; \tau) \\ - H_{k+1}(X_1, \dots, X_k, Y_j; Y_1, \dots, Y_k, Y_j; \tau)] \end{array} \right. \quad (90)$$

This system must be solved with the asymptotic condition:

$$H_1(X, Y, \tau) \sim (-\tau)^{\beta-1/2} \Psi((X-Y)(-\tau)^\beta) \quad \text{as } \tau \rightarrow -\infty \quad (91)$$

$$H_k(X_1, \dots, X_k; Y_1, \dots, Y_k; \tau) \sim \sum_{\sigma \in S^k} \prod_{m=1}^k H_1(X_{\sigma(m)}, Y_m; \tau) \quad \text{as } \tau \rightarrow -\infty \quad (92)$$

Notice that formula (90) implies that all the correlation function G_k defined in (43) became of order the order of $\prod_{m=1}^k |F_1|$ in the time scale (79) .

This problem may be also expressed in the phase space using the Wigner transform that are defined as:

$$\begin{aligned} \varphi_k(X_1, \dots, X_k; P_1, \dots, P_k; \tau) &= \frac{1}{(2\pi)^3} \int d\zeta_1 \dots d\zeta_k e^{i \sum_{j=1}^k \zeta_j P_j} \times \\ &\times H_k(X_1 - \zeta_1, \dots, X_k - \zeta_k; X_1 + \zeta_1, \dots, X_k + \zeta_k; \tau). \end{aligned}$$

Plugging this into the system (90) we obtain:

$$\begin{aligned} \frac{\partial \varphi_k}{\partial \tau} + \sum_{j=1}^k P_j \cdot \nabla_{X_j} \varphi_k &= (2\pi)^{3k} \sum_{j=1}^k \int d\zeta_j d\tilde{P}_j d\tilde{P}_{k+1} e^{i\zeta_j(P_j - \tilde{P}_j)} \times \\ &\times \left[\varphi_{k+1}(X_1, \dots, X_k, X_j - \zeta_j; P_1, \dots, \tilde{P}_j, \dots, P_k, \tilde{P}_{k+1}; \tau) - \right. \\ &\left. - \varphi_{k+1}(X_1, \dots, X_k, X_j + \zeta_j; P_1, \dots, \tilde{P}_j, \dots, P_k, \tilde{P}_{k+1}; \tau) \right]; \quad k = 1, 2, \dots \end{aligned} \quad (93)$$

The asymptotic data as $\tau \rightarrow -\infty$ are now determined by:

$$\varphi_1(X; P; \tau) = \varphi_1(P; \tau) \sim (-\tau)^{-\beta-1/2} \Phi\left(\frac{P}{(-\tau)^\beta}\right) \quad (94)$$

$$\begin{aligned} \varphi_k(X_1, \dots, X_k; P_1, \dots, P_k; \tau) &\sim \frac{1}{(2\pi)^{3k}} \sum_{\sigma \in S^k} \int d\zeta_1 \dots d\zeta_k e^{i \sum_{j=1}^k \zeta_j P_j} \times \\ &\times \prod_{m=1}^k H_1(X_{\sigma(m)} - \zeta_{\sigma(m)} - X_m - \zeta_m; \tau), \quad k > 1. \end{aligned} \quad (95)$$

Both systems, (90)-(92) and (93)-(95) are rather complicated objects to study that we do not consider in detail in this paper. However, the solution of this problem should provide a clear description on how the transition from the kinetic regime to the quantum dominated and highly correlated regime takes place.

It is interesting to compute the time for which the correlations appear in physical variables. Using (79), we obtain that such scale is given by:

$$T^* - t = \frac{2m\lambda^2}{\hbar} \left(\frac{a\lambda^2}{d^3} \right)^{-\frac{4\beta}{2\beta+1}}$$

where T^* is the time at which the Uehling Uhlenbeck equation blows up in the original physical units. The range of physical moments that would be described by the systems above (“in the boundary layer”) is

$$p \sim \frac{\hbar}{\lambda} \left(\frac{a\lambda^2}{d^3} \right)^{\frac{2\beta}{2\beta+1}}.$$

Finally the correlation lengths in this boundary layer is

$$x \sim \lambda \left(\frac{d^3}{a\lambda^2} \right)^{\frac{2\beta}{2\beta+1}}.$$

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